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Packet Radio Temporary Note No. 35

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Aloha System with Capture in Multipath Environment

The Aloha System was first described and its steady-state performance analyzed by N. Abramson.[1] [2] Its extension to a system with and without slots and capture was given by L. Roberts.[3] The following is a further extension of the Aloha System to an urban environment where mutlipath transmissions are encountered.

Model:

The system considered here is a relay station with a number of mobile terminals around it. The following assumptions are made:

(i) The mobile terminals are uniformly distributed around the relay station within a radius R.

Prob (Terminal at radius r) = f(r)

$$= \frac{2r}{R^2}, \quad 0 \le r \le R \qquad (1)$$

(ii) New and repeated messages generated by terminals are independent, Poisson distributed with new message rate r' and total message rate R'.

(iii) Message packets are of duration T seconds.

(iv)* Signal amplitudes received at relay station due to multipath transmissions are Rayleigh distributed with[4]

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* This is a simplified RF propagation model for urban environment. The randomness of the received amplitude depends on multipath transmission, midpath obstacles and variations in terrain. If all these effects are taken together, the distribution of received amplitude can be described by an infinite sum of the products of two different log-normal distributions [7]. Or as is discussed in [4], the conditional distribution for amplitude A, given M = m, is Rayleigh with mean m.

$$f_{A|M}(a,m) = \frac{\pi}{2}\,\frac{a}{m^2}\,\exp\left[-\left(\frac{\pi}{4}\right)\left(\frac{a^2}{m^2}\right)\right], \quad a \ge 0$$

where M is log-normal.

For simplicity in the analysis, only the effect of multipath transmission is considered, thus m is set to be inversely proportional to r^2.

Prob (Amplitude = a)

$$= p_A(a)$$

$$= \frac{r^4 a}{m_o} \exp\left(-\frac{r^4 a^2}{2m_o}\right) \quad , \; a \geq 0 \qquad (2)$$

where

$m_o \propto$ (mean amplitude)^2

r = distance of mobile terminal from relay station.

(v) The receiver at the relay station has a capture ratio of α. Whenever

$$\frac{a_s}{a_i} > \alpha \quad ,$$

a_s is captured by the receiver.

a_s = received amplitude of intended signal.

a_i = received amplitude of interfering signal.

Analysis:

The object of this analysis is to find the channel utilization or throughput $r'T$ of such a system.

First we will find the probability that, given a message packet and an interfering packet are present, the interference effect due to the interfering packet is negligible.

Let

A_s = amplitude of message packet at relay station.

A_i = amplitude of interfering packet at relay station.

α = receiver capture ratio defined at (v).

Define another random variable Z by

$$Z = A_s - \alpha A_i \qquad (3)$$

The interfering packet will have negligible effect if $Z \geq 0$. Notice that the message packet and the interfering packet are distinguished arbitrarily by equation (3).

The probability density of z is

$$p_Z(z) = \int_0^\infty p_{As}(a_s)\, p_{Ai}\left(\frac{a_s - z}{\alpha}\right) da_s$$

and the distribution function of Z is

$$P_Z(z) = \int_{-\infty}^{z} p_Z(z)\, dz \qquad (4)$$

From equations (2) and (4), we obtain[5]

Prob (interference negligible)

$$= 1 - P_Z(0)$$

$$= 1 - \frac{\alpha^2}{\alpha^2 + \left(\frac{r_i}{r_s}\right)^4}$$

$$= \frac{1}{1 + \alpha^2\left(\frac{r_s}{r_i}\right)^4} \qquad (5)$$

where

r_s = distance between relay station and the terminal transmitting the message packet.

r_i = distance between relay station and the terminal transmitting the interfering packet.

Therefore, given one interfering packet is present,

$$
\begin{aligned}
\text{Prob } (Z \geq 0 \mid r_1, r_s) &= \text{Prob}\left(Z \geq 0 \mid \frac{r_s}{r_1}\right) \\
&= \frac{1}{1 + \alpha^2\left(\frac{r_s}{r_1}\right)^4}
\end{aligned}
\tag{6}
$$

If we let $y = \frac{r_s}{r_1}$, from assumption (i), we obtain the probability density function of y,[6]†

$$
f(y) = \begin{cases} y & 1 \geq y \geq 0 \\ \frac{1}{y^3} & y \geq 1 \end{cases}
\tag{7}
$$

From equations (6) and (7)

Prob (interference negligible | one interfering packet)

$$
= \int_0^\infty \text{Prob } (Z \geq 0 \mid y)\, f(y)\, dy
\tag{8a}
$$

$$
= \left(\frac{1}{2\alpha} - \frac{\alpha}{2}\right) \tan^{-1} \alpha - \frac{\alpha\pi}{4} + \frac{1}{2}
\tag{8}
$$

Define capture ratio CR = 20 $\log_{10} \alpha$, then $\alpha = 1$ corresponds to the unrealistic case of CR = 0 dB, and $\alpha = \infty$ corresponds to the no capture case. From (8)

$\alpha = 0$ gives

Prob (interference negligible | one interfering packet)

$$= \frac{1}{2}$$

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† Given $f_X(x)$, $f_Y(y)$, and $X \geq o$, $Y \geq o$ then

$f_{Y/X}(z) = \int_o^\infty x f_X(x) f_Y(xz) dx, \quad o < z < \infty$.

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For $\alpha = \infty$, (8a) gives

Prob (interference negligible | one interfering packet)

$= 0$

Finally if we let

q_n = Prob (interference negligible | n interfering packet)

then

$$0 < q_n < q_{n-1} < \quad \cdots \quad < q_1 < 1 \quad ,$$

where q_1 is given by equation (8).

Since that more interfering packets are present means higher interfering power and the message packet is less likely to be captured by the receiver.†

A neat expression for q_n, $n \geq 2$ have not been found, so an upper bound and a lower bound for the channel utilization is sought.

Prob (no interference in m slots)

= Prob (no interfering packet in m slots)

$+\sum_{n=1}^{\infty}$ Prob (interfering negligible | n interfering packets in m slots)

Prob (n interfering packets)

$$= e^{-mR'T} + \sum_{n=1}^{\infty} \frac{(mR'T)^n}{n!} e^{-mR'T} q_n$$

† In L. Robert's analysis, the situation that more than one terminal outside $R\sqrt{\beta}$ are transmitting at the same time and having a total interfering power greater than the given message packet causing a conflict is not taken into account. Hence

$$q = \frac{\beta}{\gamma m}(1 - e^{-\gamma m}) + (1 - \beta)e^{-\gamma m}$$

only represents an upper bound on the probability of no conflict. Consequently, $\sigma = \gamma q$ is an upper bound on the channel utilization.

Where m = 1 for slotted Aloha and m = 2 for Aloha without slot

Letting $S = \gamma' T$ and $G = R' T$, we obtain

$$S = G\left(e^{-mG} + q_1 \, mGe^{-mG}\right)$$

$$= \quad Ge^{-mG}\left(1 + q_1 mG\right) \tag{9}$$

and maximum of S occurs at

$$mG = \frac{\left(2q_1 - 1\right) + \sqrt{4q_1^{\,2} + 1}}{2q_1}, \quad 0 \le q_1 \le \tfrac{1}{2} \tag{10}$$

for an upper bound, set

$$q_n = q_1, \; n \ge 2$$

we obtain:

$$S = \quad G\left[e^{-mG} + q_1\left(1 - e^{-mG}\right)\right] \tag{11}$$

For small α, S of Eq. (11) is an increasing function of G. True maximum exists only for large α. So it is considered not to be a tight upper bound.

<u>**Results:**</u>

Both Eq. (9) and (11) converge to $S = \quad Ge^{-mG}$

as $\alpha \to \infty$, by setting m = 2, we can obtain the Aloha system described by N. Abramson.

For the Aloha system without slot in multipath environment, m is set equal to 2.

Eq. (9) was plotted in Figure 1.

Lower bound was plotted using Eq. (9) and (10) and is given as curve B (in Figure 2). Since Eq. (11) is not a tight upper bound, values of G calculated from Eq. (10) were used in Eq. (11) to obtain an upper bound. This is given as curve A in Figure 2.

From L. Roberts' equation, setting $m = 2$ gives us the Aloha without slot.

$$\sigma_a = \frac{\beta}{2}(1 - e^{-2\gamma_a}) + (1 - \beta)\,\gamma_a e^{-2\gamma_a}$$

and

$$\sigma_{max} = \frac{1}{2}\left[\beta + (1 - \beta)e^{-\gamma_1}\right]$$

where

$$\gamma_1 = \frac{1}{1 - \beta} \quad ,$$

and

$$\beta = \frac{1}{\alpha^2} \quad ,$$

where α is defined by equation (3). σ_{max} is plotted as curve C in Figure 2.

Performance of the Aloha System with capture in multipath transmission is lower bounded by curve B. Comparison for Aloha systems in environment with and without multipath transmissions cannot be made since curve C only represents an upper bound.

For $m = 1$, the Aloha with slot, curves in Figure 1 and Figure 2 will remain the same, except that the scales of S and G axis in Figure 1 and the scale of S axis in Figure 2 should be multiplied by [illegible].

REFERENCES

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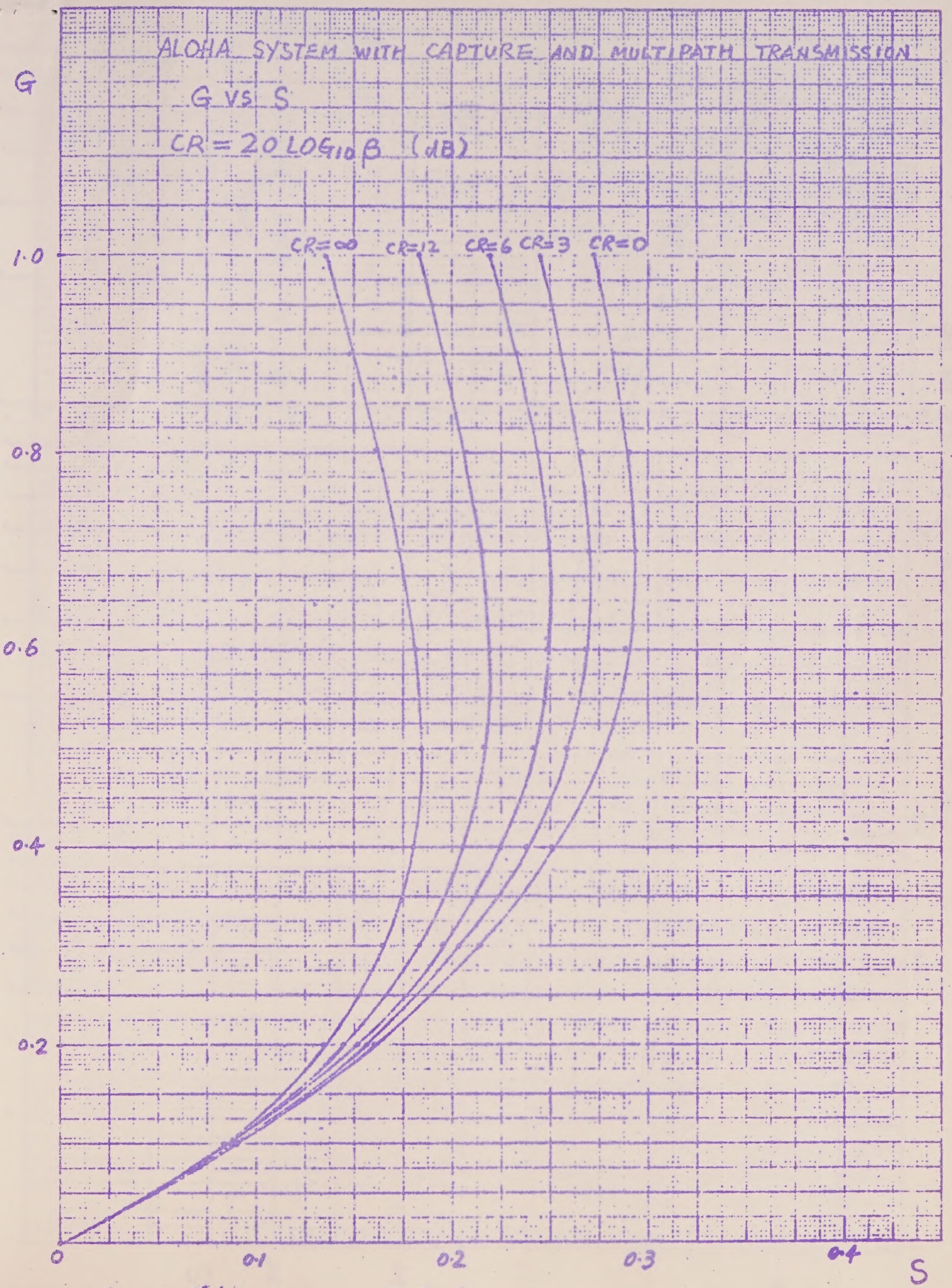
[3] L. Roberts, "ARPANET Satellite System Note 8," June 1972.

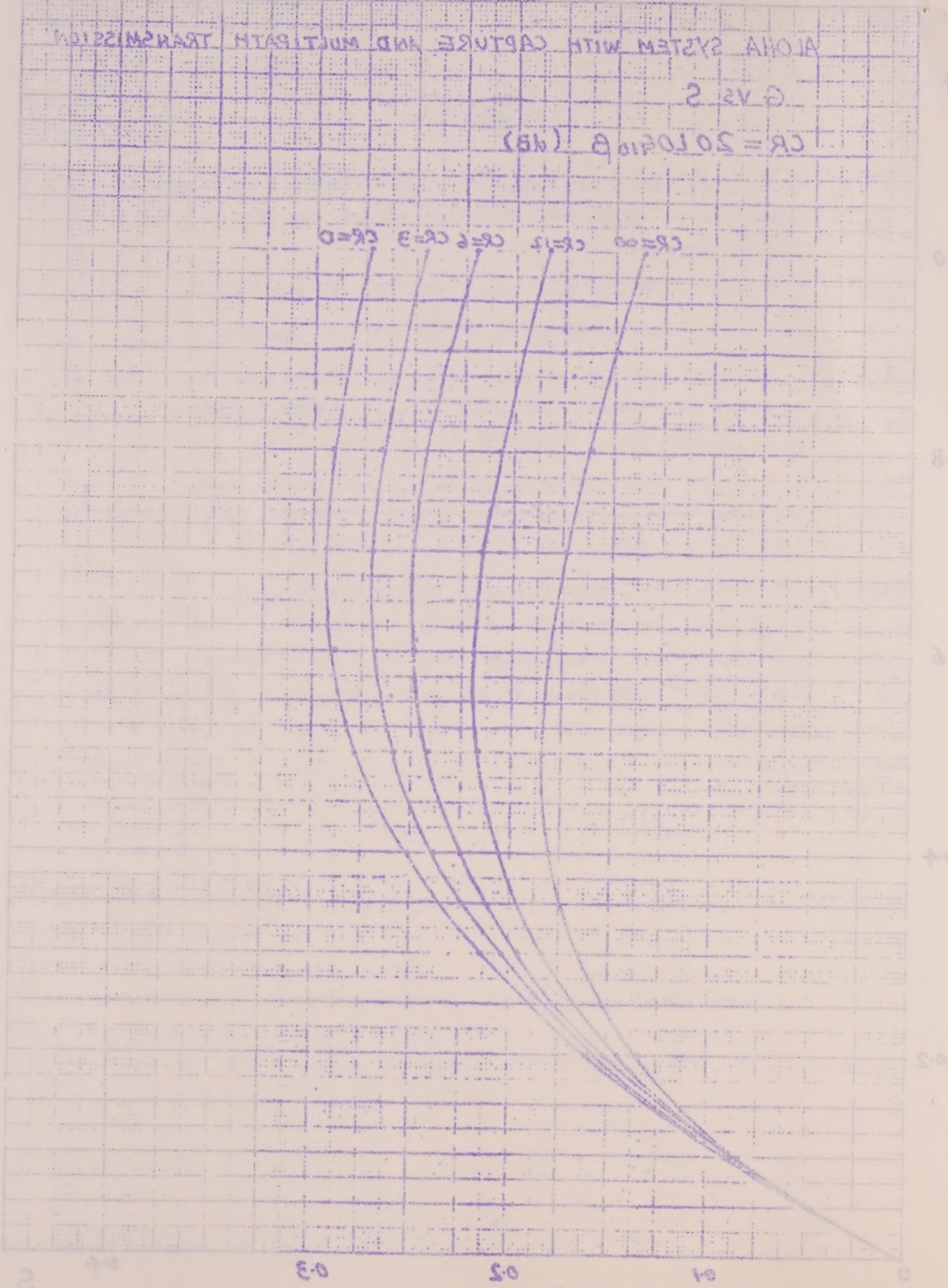
[4] L. Lundquist and M. M. Peritsky, "Cochannel Interference Rejection in a Mobile Radio Space Diversity System," IEEE Trans. on Veh. Tech., Vol. VT20, No. 3, August 1971.

[5] J. S. Engel, "The Effects of Cochannel Interference on the Parameters of a Small-Cell Mobile Telephone System," IEEE Trans. on Veh. Tech., Vol. VT-18, No. 3, November 1969.

[6] Hoel, Port, Stone, Introduction to Probability Theory, pp. 150-152, Houghton Mifflin Company, Boston, 1971.

[7] G.L. Turin et al, "A Statistical Model of Urban Multipath Propagation," IEEE Trans. on Veh. Tech., Vol. VT-21, No. 1, Feb. 1972.


ALOHA SYSTEM WITH CAPTURE AND MULTIPATH TRANSMISSION
G VS S
CR = 20 LOG10 β (dB)
CR=∞
CR=12
CR=6
CR=3
CR=0
G
1.0
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
S



ALOHA SYSTEM WITH CAPTURE AND MULTIPATH TRANSMISSION
G VS S
CR = 20 LOG10 β (dB)
CR=∞
CR=12
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1.0
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
G
S

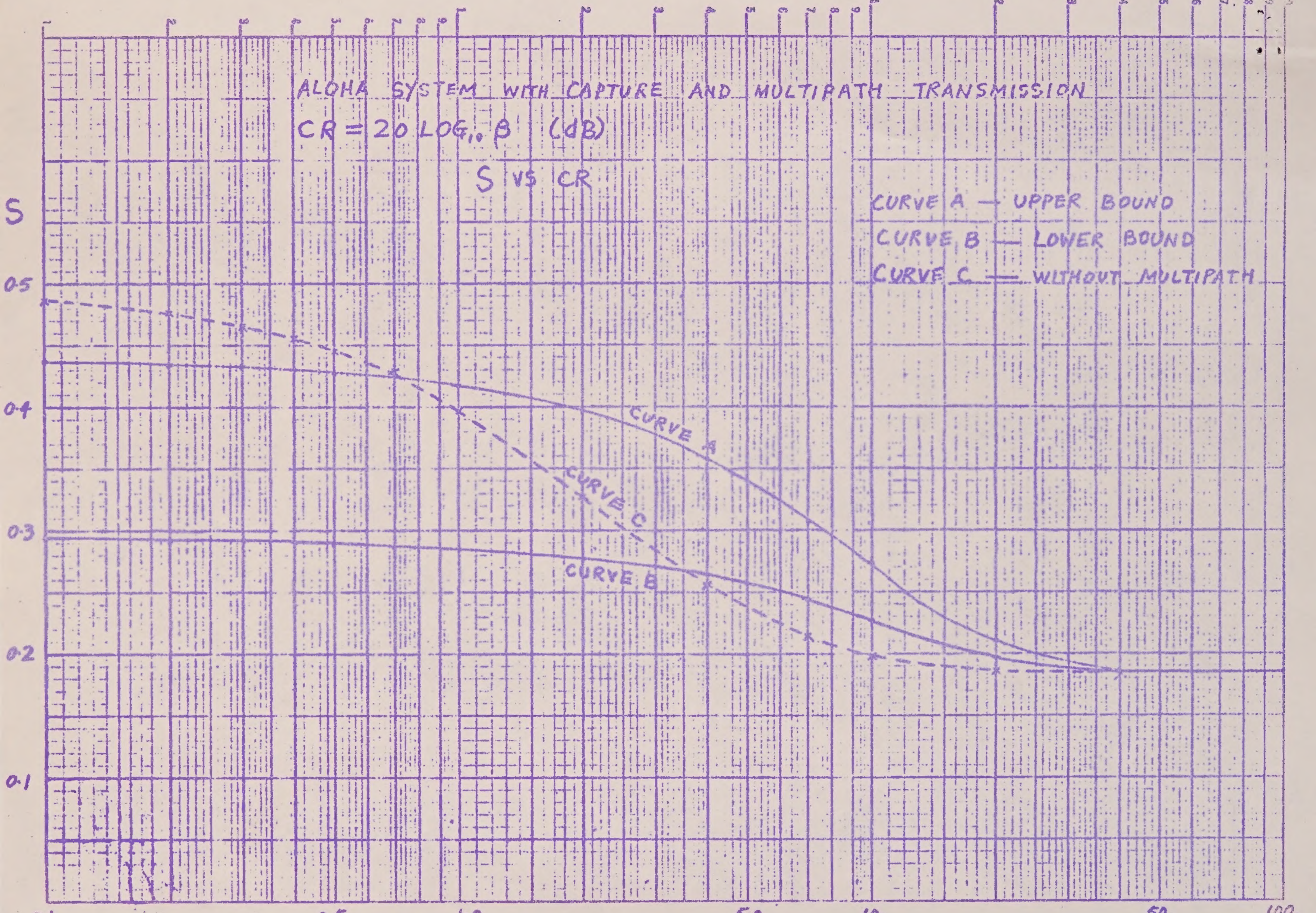
ALOHA SYSTEM WITH CAPTURE AND MULTIPATH TRANSMISSION
CR = 20 LOG₁₀ β (dB)
S VS CR
CURVE A — UPPER BOUND
CURVE B — LOWER BOUND
CURVE C — WITHOUT MULTIPATH
CURVE A
CURVE C
CURVE B
S
0.5
0.4
0.3
0.2
0.1
0.1
0.5
1.0
5.0
10.
50.
100.
CAPTURE RATIO CR IN DB

FIG. 2

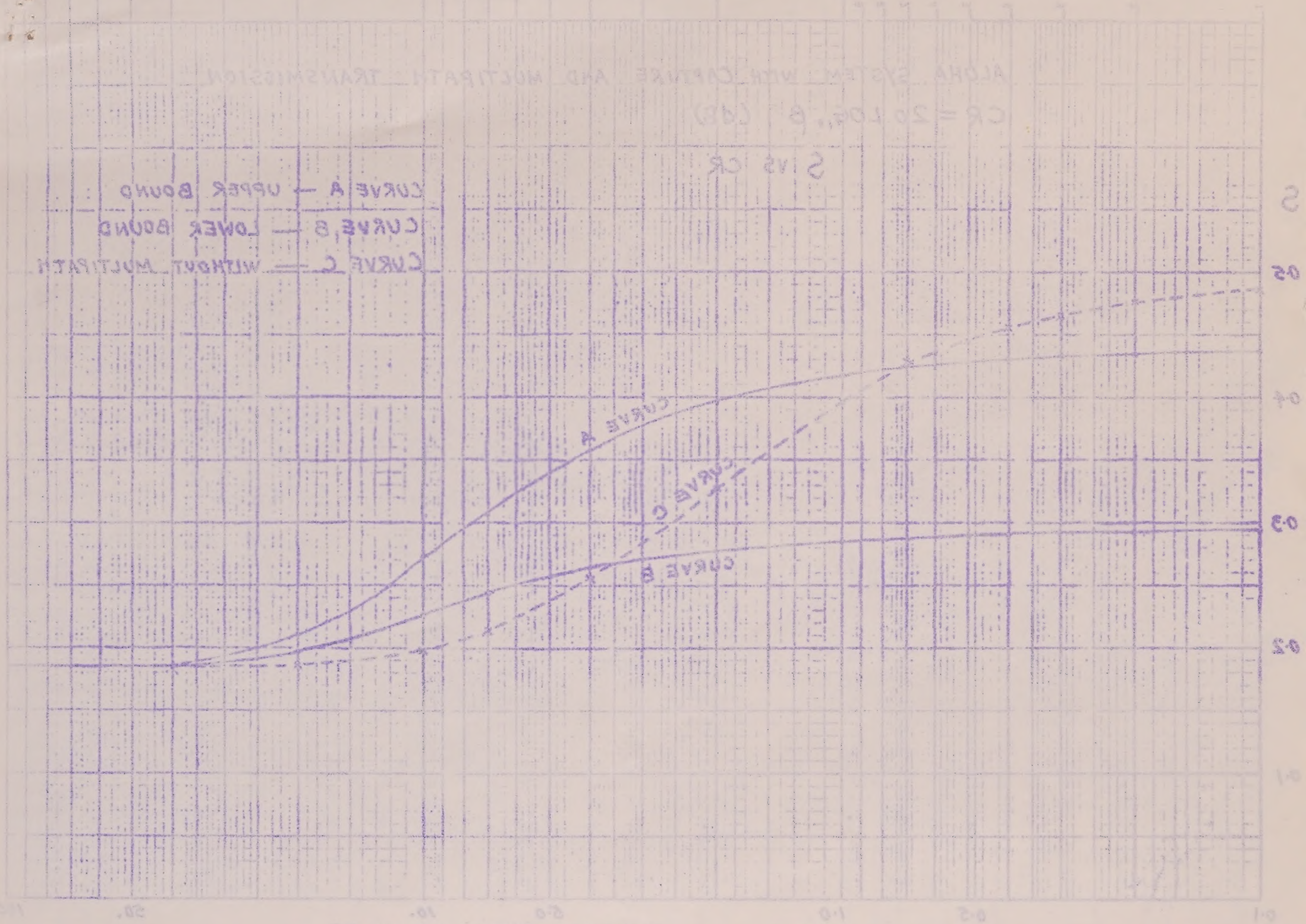
ALOHA SYSTEM WITH CAPTURE AND MULTIPATH TRANSMISSION
CR = 20 LOG₁₀ β (dB)
S VS CR
S
CURVE A — UPPER BOUND
CURVE B — LOWER BOUND
CURVE C — WITHOUT MULTIPATH
CURVE A
CURVE B
CURVE C
0·5
0·4
0·3
0·2
0·1
0·1
0·5
1·0
5·0
10.
50.
100.
CAPTURE RATIO CR IN DB

FIG. 2